Chapter 11: Infinite Sequences and Series

Section 11.1: Sequences

Objective: In this lesson, you learn how to define sequences and determine their convergence or divergence using the Limit Laws, the Squeeze Theorem, boundedness, or monotonicity.

I. Infinite Sequences

Definition: A sequence

A **sequence** is a list of n numbers written in a definite order: $a_1, a_2, a_3, \dots, a_n, \dots$. The number a_1 is called the **first term**, a_2 is the **second term**, and in general, a_n is the nth term.

Remark:

- The sequence $\{a_1,\ a_2, \dots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.
- A sequence can be defined as a function whose domain is the set of all positive integers.

$f(n) = a_n,\ n=1,2,3\dots$

Example 1: Find a formula for the general term a_n of the sequence

a. $\{\overset{a_1}{5},\ 8,\ 11,\ 14,\ 17, \dots\}$ Each term is larger than the preceding term by 3

$d = +3 \quad +3 \quad +3 \quad +3 \ \dots$

so, $a_n = a_1 + d(n-1) = 5 + 3(n-1) = 5 + 3n - 3 = 3n+2$

$n=1 \Rightarrow 3\cdot 1 + 2 = 5$, $n=3, \Rightarrow 3\cdot 3 + 2 = 11$

$n=2 \Rightarrow 3\cdot 2 + 2 = 8$, $\vdots$ $\{a_n\} = \{3n+2\}_{n=1}^{\infty}$

b. $\left\{\frac{1}{1},\ \frac{2}{3},\ \frac{3}{7},\ \frac{4}{15}, \dots\right\}$

$\{a_n\} = \left\{\frac{n}{2^n - 1}\right\}$,

$n=1 \Rightarrow \frac{1}{2^1-1} = 1$

$n=2 \Rightarrow \frac{2}{2^2-1} = \frac{2}{3}$

$n=3 \Rightarrow \frac{3}{2^3-1} = \frac{3}{7}$

$\vdots$

$n = 1 \quad 2 \quad 3 \quad 4 \quad 5$

c. $\left\{0,\ 1,\ \sqrt{2},\ \sqrt{3},\ \sqrt{4}, \dots\right\}$

$\{a_n\} = \left\{\sqrt{n-1}\right\}$

Example 2: The Fibonacci sequence is defined recursively by

$$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3.$$

Find the first 8 terms.

$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

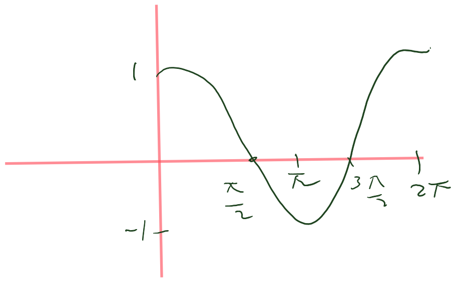
Example 3: Write out the first few terms of the sequence $\{\cos n\pi\}_{n=2}^{\infty}$ $= \{\cos(n+1)\pi\}_{n=1}^{\infty}$

$n=2 \Rightarrow \cos 2\pi = 1$

$n=3 \Rightarrow \cos 3\pi = -1$

$n=4 \Rightarrow \cos 4\pi = 1$

$n=5 \Rightarrow \cos 5\pi = -1$



$\{\cos n\pi\}_{n=2}^{\infty} = \{1, -1, 1, -1, 1, -1, \ldots\}$

$= \{(-1)^{n+1}\}_{n=1}^{\infty} = \{(-1)^n\}_{n=0}^{\infty} = \{(-1)^{n-1}\}_{n=1}$

Example 4: Find a formula for the general term a_n of the sequence

$$\left\{\frac{1}{5}, \frac{-2}{25}, \frac{6}{125}, \frac{-24}{625}, \frac{120}{3125}, \ldots\right\}$$

$5^2 \quad 5^3 \quad 5^4 \ldots$

$a_n = \left\{\frac{(-1)^{n-1} n!}{5^n}\right\}_{n=1}^{\infty}$

$n! = n(n-1)(n-2)*\cdots*2*1$

$0! = 1$

$1! = 1$

$2! = 2*1 = 2$

$3! = 3*2*1 = 6$

$4! = 4*3*2*1 = 24$

$5! = 120$

Presenting sequences

Example 5: Graph the following sequence

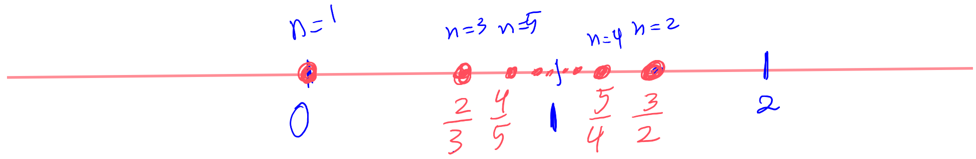
$$\left\{1+\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$$

$1-\frac{1}{3}$ $\quad 1-\frac{1}{5}$ $\quad 1+\frac{1}{4}$

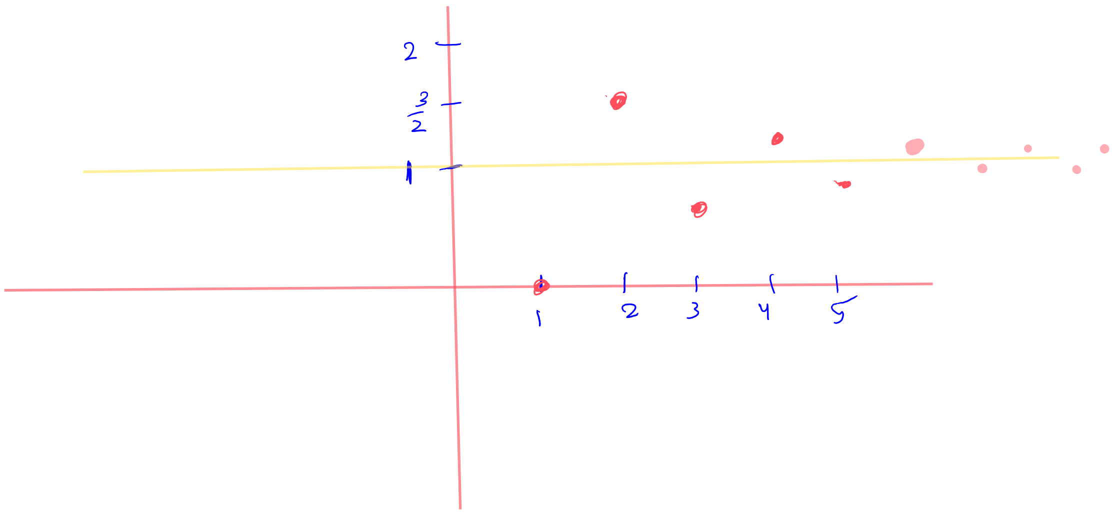
i. Using the Real Line:

$n=1$ $\quad n=2$ $\quad n=3$ $\quad n=4$ $\quad n=5$

$0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}$



ii. Using the $xy-$coordinate : since a sequence is a function whose domain is the set of positive integers, its graph consists of points with coordinates $(1, a_1), (2, a_2), \ldots, (n, a_n), \ldots$.



II. The Limit of a Sequence

We can talk about a limit L of a sequence.

Definition: Limit of a sequence

A sequence $\{a_n\}$ has the limit L and we write

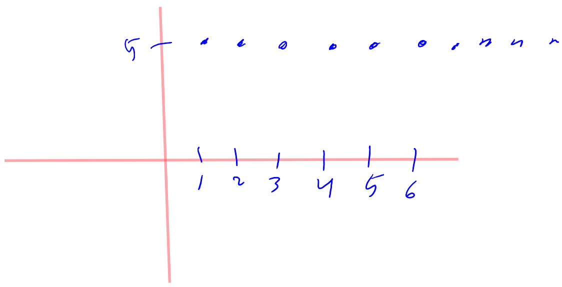
$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty,$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If the limit $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or it is **convergent**). Otherwise, we say the sequence **diverges** (or it is **divergent**).

Example 6: Does the sequence converge?

a. $a_n = \{5\}$ $= 5, 5, 5, 5, \ldots\ldots$

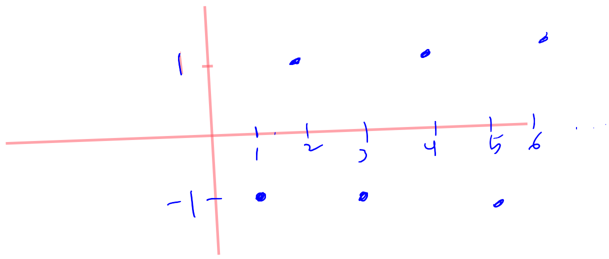
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 5 = 5$



b. $a_n = \{(-1)^n\}_{n=1}^{\infty}$ $= -1, 1, -1, 1, -1, 1, -1 \ldots\ldots$

$\lim_{n\to\infty} a_n$ does not exists.

divergent



c. $a_n = \left\{\sin\left(\frac{n\pi}{2}\right)\right\}$ $= \sin\left(\frac{\pi}{2}\right), \sin\left(\frac{2\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right), \sin\left(\frac{4\pi}{2}\right), \ldots$

$= 1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{n\pi}{2}\right) =$ D.N.E

divergent

Since the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. Thus, we have the following theorem.

Theorem 1

> If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

$$\boxed{\lim_{x\to\infty} c = c}$$

Example 7: Does the sequence converge?

a. $a_n = \left\{\frac{2n}{n+3}\right\} = \frac{2}{4}, \frac{4}{5}, \frac{6}{6}, \frac{8}{7}, \frac{10}{7}, \frac{12}{9}, \frac{14}{10}$

1st $f(x) = \frac{2x}{x+3}$

$$\lim_{x\to\infty} \frac{2x}{x+3} \quad \frac{\infty}{\infty} \text{ derivative} = \lim_{x\to\infty} \frac{2}{1} = 2$$

$$\lim_{x\to\infty} \frac{2x^1}{x^1+3} = \lim_{x\to\infty} \frac{2\cancel{x}}{\cancel{x}} = 2$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{n+3} = 2$$

Convergent

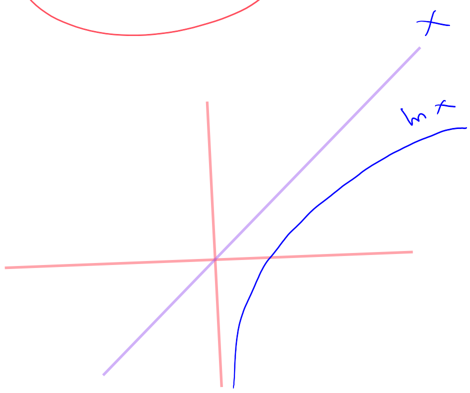
b. $a_n = \left\{\frac{n}{\sqrt{1+n}}\right\} \to f(x) = \frac{x}{\sqrt{1+x}}$ $\qquad \frac{99}{\sqrt{100}} = \frac{99}{10}$

$$\lim_{n\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} = \lim_{n\to\infty} \frac{a_n x^n}{b_m x^m} = \begin{cases} \frac{a_n}{b_m} & n = m \\ 0 & m > n \\ \infty & n > m \end{cases}$$

$$\lim_{x\to\infty} \frac{x}{\sqrt{1+x}} \quad \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{\frac{1}{2\sqrt{x+1}}}$$

$$= \lim_{x\to\infty} 2\sqrt{x+1} = \infty$$

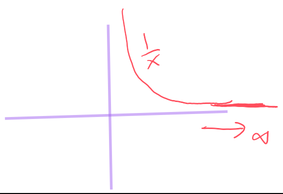
Divergent



c. $a_n = \left\{\frac{\ln n}{n}\right\} \to$ 1st $f(x) = \frac{\ln x}{x}$

$$\lim_{x\to\infty} \frac{\ln x}{x} \quad \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x} = 0$$



$\ln \infty = \infty$

$\frac{1}{\infty} = 0$

d. $a_n = \{\sqrt[n]{n}\}$ $= \sqrt[1]{1} = 1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \sqrt[5]{5}, \dots$

let $a_n = f(n)$

$$f(x) = \sqrt[x]{x^1} = x^{\frac{1}{x}}$$

$$\lim_{x \to \infty} f(x) = \infty^{\frac{1}{\infty}} = \infty^0$$

let $y = x^{\frac{1}{x}}$

$$\ln y = \ln x^{\frac{1}{x}} = \frac{1}{x} \ln x$$

$\ln A^n = n \ln A$

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln x}{x} \quad \frac{\infty}{\infty} \overset{\text{From C}}{=} 0$$

$$\ln \lim_{x \to \infty} y = 0$$

$\lim_{x \to a} \ln f(x) = \ln \lim_{x \to a} f(x)$

$$\cancel{e}^{\cancel{\ln} \lim_{x \to \infty} y} = e^0 = 1$$

$$\lim_{x \to \infty} y = 1$$

$$\lim_{x \to \infty} x^{\frac{1}{x}} = 1$$

a_n is convergent

Limit Laws for sequence

The Limit Laws for functions also hold for the limits of sequences and their proofs are similar.

Limit Laws for sequence

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$.
4. $\lim_{n\to\infty} c = c$.
5. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
6. $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, if $\lim_{n\to\infty} b_n \neq 0$.
7. $\lim_{n\to\infty}(a_n)^b = \left[\lim_{n\to\infty} a_n\right]^b$, if $b > 0$ and $a_n > 0$.

Example 8: Does the sequence converge?

$$a_n = \left\{\frac{3n}{n+2} + \frac{n^2}{n^2+1}\right\}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left\{\frac{3n}{n+2} + \frac{n^2}{n^2+1}\right\}$$

$$= \lim_{n\to\infty}\frac{3n}{n+2} + \lim_{n\to\infty}\frac{n^2}{n^2+1}$$

$$= \lim_{n\to\infty}\frac{3\cancel{n}}{\cancel{n}} \to 3 + \lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}} \to 1$$

$$= 3 + 1 = 4$$

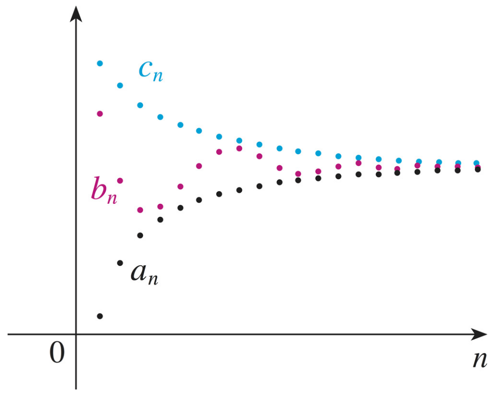
convergent.

The Squeeze Theorem

Theorem 2

The Squeeze Theorem also hold for sequences: If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

$$\lim_{n\to\infty} b_n = L.$$



Example 9: Does the sequence converge?

$$a_n = \frac{n!}{n^n}$$

$1, \frac{2}{4}, \frac{6}{3^3}, \frac{24}{4^4}, \ldots$ $\qquad 0 \leq a_n$ for all $n \geq 1$

$n=1 \Rightarrow 1 \leq 1$

$n=2 \Rightarrow \frac{2}{4} = \frac{1}{2} \leq \frac{1}{2}$

$n=3 \Rightarrow \frac{6}{27} = \frac{2}{3\cdot 3} \leq \frac{1}{3}$

$n=4 \Rightarrow \frac{\cancel{24}\,6}{\cancel{4}\cdot 4\cdot 4\cdot 4} = \frac{3}{2*4*4} \leq \frac{1}{4}$

$\therefore a_n \leq \frac{1}{n}$ for all n.

So, $0 \leq a_n \leq \frac{1}{n}$

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} \frac{1}{n}$$

$$0 \leq \lim_{n\to\infty} a_n \leq 0$$

So, $\lim_{n\to\infty} a_n = 0$.

Theorem 3

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example 10: Calculate $\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+1}$

$$\lim_{n\to\infty} \left|\frac{(-1)^n n^2}{n^2+1}\right| = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{\cancel{n^2}\,1}{\cancel{n^2}} = 1 \neq 0$$

So, $\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+1}$ D.N.E

$\cos n\pi$
$n=1 \Rightarrow \cos\pi = -1$
$n=2 \Rightarrow \cos 2\pi = 1$
$n=3 \Rightarrow \cos 3\pi = -1$
1
-1
$\vdots$

$-1 < \cos x < 1$

$|\cos x| \leq 1$

Example 11: Calculate $\lim_{n\to\infty} \frac{\cos(n\pi)}{n}$

$$\lim_{n\to\infty} \frac{\cos(n\pi)}{n} = \lim_{n\to\infty} \frac{(-1)^n}{n} \ ?$$

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

So, $\lim_{n\to\infty} \frac{\cos(n\pi)}{n} = 0$

$$0 \leq \lim_{n\to\infty} \left|\frac{\cos n\pi}{n}\right| \leq \lim_{n\to\infty} \frac{1}{n} = 0 \longrightarrow \lim_{n\to\infty} \left|\frac{\cos n\pi}{n}\right| = 0$$

Example 12: For what values of r is the sequence $\{r^n\}$ convergent?

① If $r = 1 \Rightarrow 1^n = 1 \Rightarrow \lim_{n\to\infty} 1^n = 1$ conv.

② If $r = 0 \Rightarrow 0^n = 0 \Rightarrow \lim_{n\to\infty} 0^n = 0$ conv.

③ If $r > 1 \Rightarrow \lim_{n\to\infty} r^n = \infty$ div.

④ If $0 < r < 1 \Rightarrow \lim_{n\to\infty} r^n = 0$ conv.

⑤ If $-1 < r < 0 \Rightarrow \lim_{n\to\infty} r^n = 0$ conv.

⑥ If $r = -1 \Rightarrow \lim_{n\to\infty} (-1)^n =$ div.

⑦ If $r < -1 \Rightarrow \lim_{n\to\infty} (-2)^n =$ div.

$\lim_{n\to\infty} \left(\frac{-1}{2}\right)^n = \lim_{n\to\infty} \frac{(-1)^n}{2^n}$

$\lim_{n\to\infty} \left|\frac{(-1)^n}{2^n}\right| \to 0$

$\lim_{n\to\infty} |(-1)^n| = \lim_{n\to\infty} 1 = 1 \neq 0$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \\ \text{div} & \text{o.w} \end{cases}$$

Example 13: For what values of p is the sequence $\left\{\frac{1}{n^p}\right\}$ convergent?

① If $p = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{n^0} = \lim_{n\to\infty} 1 = 1$ conv.

② If $p > 0 \Rightarrow \lim_{n\to\infty} \frac{1}{n^p} = 0$ conv.

③ If $p < 0 \Rightarrow \lim_{n\to\infty} \frac{1}{n^p} = \infty$ div.

$\frac{1}{n^{-2}} = n^2$

$$\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} 1 & p = 0 \\ 0 & p > 0 \\ \text{div} & p < 0. \end{cases}$$

$p < 0 \Rightarrow \frac{1}{n^{-2}} = n^2 \to \lim_{n\to\infty} n^2 = \infty$

III. Monotonic and Bounded Sequences

Definition

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is **monotonic** if it is either increasing or decreasing.

If a_n is increasing $\Rightarrow$ $a_1 < a_2 < a_3 < a_4 < \cdots < a_n < \cdots$

If a_n is decreasing $\Rightarrow$ $a_1 > a_2 > a_3 > a_4 > \cdots > a_n > \cdots$

Example 14: Determine whether the sequence is increasing or decreasing.

a. $a_n = 1 + \dfrac{1}{n}$ compare with a_{n+1}

$a_n = 1 + \frac{1}{n}$ $\quad a_{n+1} = 1 + \frac{1}{n+1}$

$n < n+1$ $\quad n \geq 1$

$\frac{1}{n} > \frac{1}{n+1}$ $\quad n \geq 1$

$1 + \frac{1}{n} > 1 + \frac{1}{n+1}$

$a_n > a_{n+1} \rightarrow$ decreasing

b. $b_n = 1 - \dfrac{1}{n}$ $\quad b_{n+1} = 1 - \frac{1}{n+1}$

$n < n+1$

$\frac{1}{n} > \frac{1}{n+1}$

$-\frac{1}{n} < \frac{-1}{n+1}$

$1 - \frac{1}{n} < 1 - \frac{1}{n+1} \rightarrow b_n < b_{n+1}$ increasing

c. $c_n = 1 + \dfrac{(-1)^n}{n}$

$\overset{1}{0} \leq \overset{2}{1 + \frac{1}{2}} \geq \overset{3}{1 - \frac{1}{3}} \leq \overset{4}{1 + \frac{1}{4}} \geq \overset{5}{1 - \frac{1}{5}},$

For n even, $c_n = 1 + \frac{1}{n} \rightarrow$ from (a) It is (decreasing)

For n odd, $c_n = 1 - \frac{1}{n} \rightarrow$ from (b) It is (increasing)

$\Big\}$ neither.

Definition

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M$$

for all $n \geq 1$. It is **bounded below** if there is a number m such that

$$m \leq a_n$$

for all $n \geq 1$. If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Note the following:

i. A sequence can be bounded above but not below.

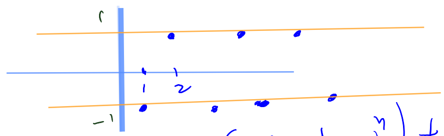
ii. Not every bounded sequence is convergent.

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

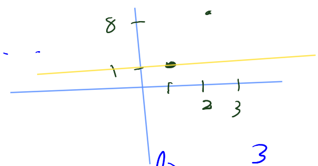
Example 15: Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

a. $a_n = (-1)^n$ $= -1, 1, -1, 1, -1, 1, \dots$



- a_n is bounded above by 1 below by -1
- is not monotonic. $\lim_{n\to\infty} (-1)^n$ div. because $\left(\lim_{n\to\infty} |(-1)^n| \neq 0\right)$

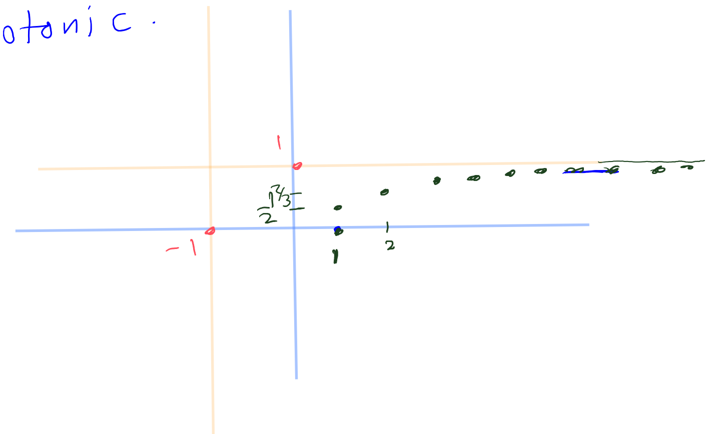
b. $a_n = n^3$, $n \geq 1$, $a_n = 1, 2^3, 3^3, 4^3, \dots$

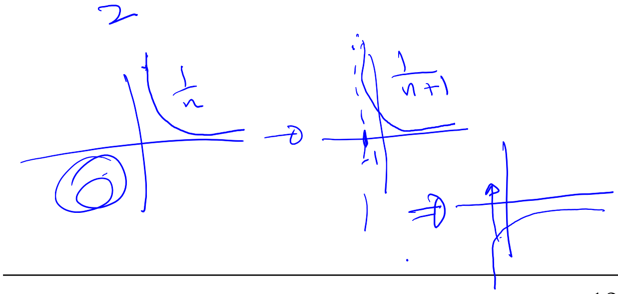


- bounded below by 1
- a_n is increasing ↗ monotonic. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^3 = \infty$ div.

c. $a_n = \dfrac{n}{n+1}$ $= \dfrac{n+1-1}{n+1} = 1 - \dfrac{1}{n+1}$, $n \geq 1$

- a_n is increasing ↗ monotonic.
- $\frac{1}{2} \leq a_n \leq 1$ bounded





$a_n = \dfrac{n}{n+1}, n \geq 1 \to b_n = \dfrac{n+11}{n+12}$ $n \geq -10$

Example 16: Determine whether the sequence $a_n = ne^{-n}$ is increasing or decreasing. Is the sequence bounded?

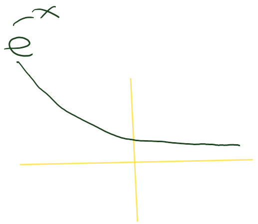
$f = g \cdot h$

$f' = g \cdot h' + h g'$

let $f(x) = x e^{-x}$.

$f'(x) = -x e^{-x} + e^{-x}$

$= e^{-x}(1-x) = 0$



$e^{-x} \neq 0$ or $1 - x = 0$

$x = 1$

sign of $f'(x)$

$+ + + + + + + \quad | \quad - - - - - - -$

$0 \quad 1 \quad 2$

$f'(0) = e^{-0}(1-0) = +$

$f'(2) = e^{-2}(1-2) = -$

f is $\searrow$ on $[1, \infty)$

a_n is decreasing $\searrow$ for $n \geq 1$

$a_n = n e^{-n}$.

$n = 1 \Rightarrow a_1 = 1 \cdot e^{-1} = e^{-1} \Rightarrow$ the a_n is bounded above by e^{-1}

$$\lim_{n\to\infty} n e^{-n} \overset{\infty \cdot e^{-\infty} = \infty \cdot 0}{=} \lim_{n\to\infty} \frac{n}{e^n} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{1}{e^n} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

bounded below by 0

By MST, a_n is convergent seq.

Example 17: Find the limit of $\left\{\sqrt{3},\ \sqrt{3\sqrt{3}},\ \sqrt{3\sqrt{3\sqrt{3}}},\ ...\right\}$.

$$a_1 = \sqrt{3} = 3^{1/2}$$

$$a_2 = \sqrt{3\sqrt{3}} = \sqrt{3 \cdot 3^{1/2}} = \sqrt{3^{3/2}} = 3^{3/4} \quad 3 = 2^2 - 1,\ 4 = 2^2$$

$$a_3 = \sqrt{3\sqrt{3\sqrt{3}}} = \sqrt{3\sqrt{3 \cdot 3^{1/2}}} = \sqrt{3\sqrt{3^{3/2}}} = \sqrt{3 \cdot 3^{3/4}} = \sqrt{3^{7/4}} = 3^{7/8} \quad 7 = 2^3 - 1,\ 8 = 2^3$$

$$a_4 = \quad = 3^{15/16} \quad 15 = 2^4 - 1,\ 16 = 2^4$$

$$\vdots$$

$$a_n = 3^{\frac{2^n - 1}{2^n}} = 3^{\frac{2^n}{2^n} - \frac{1}{2^n}} = \left\{ 3^{1 - \frac{1}{2^n}} \right\}_{n=1}^{\infty}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} 3^{1 - \frac{1}{2^n}}$$

$$= 3^{1 - \frac{1}{\lim_{n \to \infty} 2^n}}$$

$$= 3^{1 - \frac{1}{\infty}}$$

$$= 3^{1 - 0} = 3.$$